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THREE ALGEBRAIC NOTES.

By DR. L. E. DICKSON.

I. TRANSFORMATION OF A SOLVABLE QUINTIC INTO ITSELF.

As well known, the following quintic equation

$$y^5+py^3+\tfrac{1}{5}p^2y+r=0 \tag{1}$$

may be solved algebraically by making the substitution

$$y=z-\tfrac{1}{5}p/z,$$

quite analogous to Cartan's solution of a reduced cubic equation.

We here consider another property of (1), which may be written

$$y\{(y^2+\tfrac{1}{2}p)^2-\tfrac{1}{20}p^2\}+r=0. \tag{1'}$$

This form suggests the following quadratic transformation

$$y^2+\tfrac{1}{2}p=w,\ y=\surd(w-\tfrac{1}{2}p).$$

After rationalization, (1′) then becomes

$$(w-\tfrac{1}{2}p)(w^2-\tfrac{1}{20}p^2)^2=r^2,$$

a solvable quintic containing al powers $\leqq 5$ of w. To remove the term w^4, set $w=x+\frac{1}{10}p$. We thus obtain

$$(x-\tfrac{2}{5}p)(x^2+\tfrac{1}{5}px-\tfrac{1}{25}p^2)^2=r^2,$$

$$x^5-\tfrac{1}{5}p^2x^3+\tfrac{1}{125}p^4x-\frac{2}{5^5}p^5-r^2=0, \tag{2}$$

a quintic lacking the term x^2 as well as x^4. Setting

$$Y=ax,\ P=-\tfrac{1}{5}p^2a^2,\ R=-\frac{2}{5^5}p^5a^5-r^2a^5,$$

equation (2) takes the following form, analogous to (1),

$$Y^5+PY^3+\tfrac{1}{5}P^2Y+R=0. \tag{3}$$

Now equations (1) and (3) are identical in exactly two cases:

$$a=\sqrt{\frac{-5}{p}},\ r=\sqrt{\left(\frac{-p}{5}\right)^5};\ a=-\sqrt{\frac{-5}{p}},\ r=-2\sqrt{\left(\frac{-p}{5}\right)^5}.$$

Theorem. *According as* $r=[\sqrt{(-p/5)}]^5$ *or* $-2[\sqrt{(-p/5)}]^5$, *the substitution* $y^2=Y\sqrt{(-p/5)}-\frac{2}{5}p$, *or* $y^2=-Y\sqrt{(-p/5)}-\frac{2}{5}p$ *transforms the quintic equation* (1) *into itself.*

In the second case, $\frac{r^2}{4}+\left(\frac{p}{5}\right)^5=0$, so that the quintic has two pairs of equal roots. In fact (Dickson's *College Algebra*, page 189), the roots are

$$2\sqrt{(-p/5)},\ \text{two each } \alpha\sqrt{(-p/5)},\ \text{two each } \beta\sqrt{-p/5}),$$

where $\alpha=\epsilon+\epsilon^4$, $\beta=\epsilon^2+\epsilon^3$ are the roots of $w^2+w-1=0$; also $\epsilon^5=1$.

II. Reducibility of a Reciprocal Sextic Equation.

The general reciprocal equation of the sixth degree may be written

$$x^6-px^5+qx^4-rx^3+qx^2-px+1=0. \tag{4}$$

As known, its solution may be made to depend upon a cubic and a quadratic equation; thus, set

$$x+\frac{1}{x}=z,\ \text{whence}\ x^2+\frac{1}{x^2}=z^2-2,\ x^3+\frac{1}{x^3}=z^3-3z. \tag{5}$$

Dividing (4) by x^3, we then obtain

$$z^3-pz^2+(q-3)z+2p-r=0. \tag{6}$$

The question here proposed is the determination of the conditions under which (4) has a factor of specified form and with coefficients in an initially given domain of rationality. Some examples will make clear what is meant by a *domain*. It is a familiar theorem that, when p is a prime number, $x^{p-1}+\dots+x^2+x+1$ is algebraically irreducible, not being expressible as a product of two factors each of the form $f=ax^n+bx^{n-1}+\dots+t$, where $1\leqq n<p-1$, and $a, b, \dots, t$ are *integers*. It readily* follows that there is no factor f with *rational* coefficients. Hence $x^{p-1}+\dots+x^2+x+1$ is irreducible in the *domain of all rational numbers*. However, it may become reducible in a larger domain. Thus, for $p=5$,

$$x^4+x^3+x^2+x+1=[x^2+\tfrac{1}{2}x(1+\sqrt{5})+1][x^2+\tfrac{1}{2}x(1-\sqrt{5})+1],$$

and hence is reducible in the domain of all rational functions of $\sqrt{5}$ with integral coefficients. It becomes completely reducible into linear factors in a domain containing an imaginary fifth root of unity.

We return to the question of the reducibility of (4) in a given domain or field F containing the coefficients of p, q, r. The condition that (4) shall have a

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*For one proof, with references to the various proofs, of these two statements, see the writer's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons, 1903), page 76.

linear factor $x-x_1$ is that the cubic (6) shall have a root z_1 in the domain F such that $x^2-z_1x+1=0$ has a root x_1 in F, the latter requiring that z_1^2-4 be the square of an element of F. With one linear factor $x-x_1$ is associated a second, viz., $x-1/x_1$. The condition for at least three linear factors is that (6) shall have at least two and hence all its roots z in F, such that z^2-4 be a square in F.

From the above method of solving (4), or by undetermined coefficients, it follows that (4) has a factor of the form x^2-z_1x+1 if and only if equation (6) has a root z_1 in F. Next, if (4) has a factor x^2-kx+t, where $t\neq 0$, $t\neq 1$, it has also the factor $t^{-1}(tx^2-kx+1)=x^2-t^{-1}kx+t^{-1}$, and hence a third factor x^2-zx+1. The condition for this third factor was that (6) have a root z in F. Let this condition be satisfied. Then the quotient of (4) by x^2-zx+1 is the reciprocal quartic equation with coefficients in F:

$$x^4+(z-p)x^3+(z^2-zp-1+q)x^2+(z-p)x+1=0. \tag{7}$$

We are therefore led to consider the conditions for the factorization

$$x^4-ax^3+bx^2-ax+1=(x^2-kx+t)(x^2-kt^{-1}x+t^{-1}). \tag{8}$$

The conditions are

$$a=k(1+t^{-1}),\quad b=t+t^{-1}+k^2t^{-1}. \tag{9}$$

Eliminating* k, we obtain the reciprocal condition on t

$$t^4+(2-bt^3+(a^2-2b+2)t^2+(2-b)t+1=0. \tag{10}$$

If (10) has a root $t\neq -1$ in F, the first condition (9) determines k in F. If every root of (10) equals -1, then $a=0$, $b=-2$, $k=0$, so that the factorization (8) is evident. For the general case, we may readily determine the conditions under which (10) has a root in F. First, it is necessary that $Z\equiv t+t^{-1}$ belong to F. By (10),

$$Z^2-(2-b)Z+a^2-2b=0,\quad (Z-1+\tfrac{1}{2}b)^2=(1+\tfrac{1}{2}b)^2-a^2.$$

Hence $(1+\frac{1}{2}b)^2-a^2$ must be the square of an element l of F. Next, from $t^2-tZ+1=0$, $Z^2-4\equiv\frac{1}{4}b^2-2-a^2+l(2-b)$ must be a square in F.

To proceed otherwise, let z_1 and z_2 be the roots of $z^2-az+b-2=0$. Then

$$x^4-ax^3+bx^2-ax+1=(x^2-z_1x+1)(x^2-z_2x+1).$$

Setting $Z_1=\sqrt{(z_1^2-4)}$, $Z_2=\sqrt{(z_2^2-4)}$, the four roots are

$$\tfrac{1}{2}z_1+\tfrac{1}{2}Z_1,\quad \tfrac{1}{2}z_1-\tfrac{1}{2}Z_1,\quad \tfrac{1}{2}z_2+\tfrac{1}{2}Z_2,\quad \tfrac{1}{2}z_2-\tfrac{1}{2}Z_2.$$

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*Upon eliminating t, there results a non-reciprocal quartic for k.

The two roots of either of the quadratic factors (8) are not reciprocal. By proper choice of the signs of Z_1 and Z_2, we may assume that $\frac{1}{2}z_1+\frac{1}{2}Z_1$ and $\frac{1}{2}z_2+\frac{1}{2}Z_2$ are the roots of one of the quadratic factors (8) and the remaining two the roots of the other. Now the coefficients of a quadratic (8) belong to F if and only if the sum and product of its two roots do. But $z_1+z_2=a$ and $z_1z_2=b-2$ belong to F. The conditions are therefore that

$$Z_1+Z_2,\ z_1Z_2+z_2Z_1+Z_1Z_2,\ -z_1Z_2-z_2Z_1+Z_1Z_2$$

shall belong to F. It follows first that $Z_1Z_2\equiv 2\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$ shall belong to F, say $=-2l$. Next, $z_1Z_2+z_2Z_1$ must belong to F. Its square equals

$$2z_1^2z_2^2-4z_1^2-4z_2^2+2z_1z_2Z_1Z_2=\tfrac{1}{4}[\tfrac{1}{2}b^2-2-a^2+l(2-b)].$$

Hence $\frac{1}{2}b^2-2-a^2+l(2-b)$ must be a square in F. We have now obtained the two conditions given earlier. I remains to verify that Z_1+Z_2 belong to F. But

$$(z_1+z_2)(Z_1+Z_2)=(z_1Z_1+z_2Z_2)+(z_1Z_2+z_2Z_1),$$

$$(z_1Z_1+z_2Z_2)(z_1Z_2+z_2Z_1)=z_1z_2(Z_1^2+Z_2^2)+Z_1Z_2(z_1^2+z_2^2),\text{ in } F.$$

Consider finally the most interesting case of all, that in which the sextic (4) has an irreducible cubic factor. Call its roots κ, λ, μ. No two of the three are reciprocal, since the product of all three would then equal the third root, which would thus belong to F, making the cubic reducible. Hence κ^{-1}, λ^{-1}, μ^{-1} are the roots of a second cubic factor of (4). Hence (4) is the product of the two cubic factors

$$x^3+ax^2+bx+c,\quad x^3+\frac{b}{c}x^2+\frac{a}{c}x+\frac{1}{c}. \tag{11}$$

Equating the coefficients of the product to those of (4), we get

$$a+\frac{b}{c}=-p,\quad b+\frac{ab}{c}+\frac{a}{c}=q,\quad c+\frac{1}{c}+\frac{b^2}{c}+\frac{a^2}{c}=-r. \tag{12}$$

Substituting the value of a from the first in the last two, we get

$$b^2+b(1+pc-c^2)+pc+qc^2=0, \tag{13}$$

$$b^2(1+c^2)+2bpc+p^2c^2+c^2+c^4+rc^3=0. \tag{14}$$

Eliminating b^2, we get $bU+W=0$, where

$$U=c^4-pc^3+pc-1,\quad W=p^2c^2+c^2+c^4+rc^3-pc-pc^3-qc^2-qc^4. \tag{15}$$

Substituting $b=-W/U$ in (13) and cancelling the factor c^2, we get

$$(16)\quad c^8+1+(c^7+c)(r-2p)+(c^6+c^2)(q^2-2pr+2p^2+1-2q)$$
$$+(c^5+c^3)(r-2p)(p^2-2q+1)+c^4(2q^2+r^2+p^4+4p^2-4q-4p^2q)=0,$$

which is evidently a reciprocal equation. Making the substitution, similar to (5),

$$(17)\qquad c+\frac{1}{c}=y,\ c^2+\frac{1}{c^2}=y^2-2,\ c^3+\frac{1}{c^3}=y^3-3y,\ c^4+\frac{1}{c^4}=y^4-4y^2+2,$$

equation (16) becomes

$$(18)\qquad y^4+y^3(r-2p)+y^2(q^2-2pr+2p^2-3-2q)$$
$$+y(r-2p)(p^2-2q-2)+p^4-4p^2q+4pr+r^2=0.$$

Suppose that y and then c can be determined in F. Now

$$(19)\qquad U=(c^2-1)(c^2-pc+1).$$

If $U\neq 0$, then b and a are uniquely determined by $bU+W=0$ and the first relation (12). Next, $U=0$ if and only if $c=\neq 1$ or $c^2+1=cp$, whence $y=2$, -2, or p. If $c=1$, $W=0$ gives $r-2q+p^2-2p+2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2+bp+p+q=0$. If $c=-1$, $W=0$ gives $r+2q-2p-p^2-2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2-bp-p+q=0$. If $c^2=pc-1$, whence $y=p$, then $W=0$ gives $r+p-pq=0$, and (18) reduces to $(r+p-pq)^2=0$, while (13) and (14) reduce $b^2+2b+qc^2+pc=0$.

In general, the values of a and b follow rationally when c is found from the quartic and quadratic. In the three cases mentioned, c is given, but b requires the solution of a quadratic.

III. THE GENERAL TERM OF A RECURRING SERIES.

The usual examples under this topic lead to a generating fraction which can be resolved into partial fractions with binomial denominators, so that the general term of the expansion into series is readily found. It is otherwise with the following example (No. 4, page 148, Dickson's *College Algebra*):

$$(20)\qquad 1+6x-4x^2-40x^3-112x^4-32x^5+704x^6+\ldots$$

The generating relation is $u_n-2xu_{n-1}+4x^2u_{n-2}+8x^3u_{n-3}=0$; the generating fraction is

$$(21)\qquad (1+4x-12x^2)/D,\ D\equiv 1-2x+4x^2+8x^3$$

First method. Set $D=(1-2\alpha x)(1-2\beta x)(1-2\gamma x)$. Then

$$(22)\qquad \alpha+\beta+\gamma=1,\ \alpha\beta+\alpha\gamma+\beta\gamma=1,\ \alpha\beta\gamma=-1.$$

The discriminant $\triangle\equiv(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ is found to be -44. Now

$$\frac{1}{D}=\frac{a}{1-2\alpha x}+\frac{b}{1-2\beta x}+\frac{c}{1-2\gamma x},$$

$$1=a\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)=b\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\gamma}{\beta}\right)=c\left(1-\frac{\alpha}{\gamma}\right)\left(1-\frac{\beta}{\gamma}\right).$$

$$\therefore\ \frac{-\triangle}{D}=\frac{\alpha^2(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha)}{1-2\alpha x}+\frac{\beta^2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2}{1-2\beta x}$$

$$+\frac{\gamma^2(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha)}{1-2\gamma x}.$$

Expanding $1/(1-2\alpha x)$, ..., into series, we obtain as the coefficient of $2^r x^r$:

$$C_r=\alpha^{r+2}(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha)+\beta^{r+2}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2$$

$$+\gamma^{r+2}(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha).$$

In this the coefficient of α^{r+2} equals

$$3\beta^2\gamma^2-(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)+3\alpha^2\beta\gamma-\alpha\beta\gamma(\alpha+\beta+\gamma)+\alpha\beta^3+\alpha\gamma^3-\beta\gamma^3-\beta^3\gamma$$

$$=3\beta^2\gamma^2+\beta\gamma-\alpha^4-9\alpha-2,$$

in view of (22) and the consequent relations

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)=3,$$

$$\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)+3\alpha\beta\gamma=-5,$$

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=-1.$$

Hence, if we set $S_n=\alpha^n+\beta^n+\gamma^n$, we obtain

(23) $$C_r-S3_r-S_{r+1}-2S_{r+2}-9S_{r+3}-S_{r+6}.$$

Since α, β, γ are the roots of y^3-y^2+y+1, we have Newton's identity

(24) $$S_k-S_{k+1}+S_{k-2}+S_{k-3}=0 \quad (k\geqq 3).$$

Hence

(25) $$S_k+2S_{k-3}+S_{k-4}=0.$$

Hence (23) becomes

(26) $$C_r=10S_r+6S_{r+1}-8S_{r+2}.$$

Now C_r denoted the coefficient of $2^r x^r$ in the expansion of $44/D$. Hence the coefficient of $2^{r+2}x^{r+2}$ in the expansion of (21) is

$$\tfrac{1}{44}(C_{r+2}+2C_{r+1}-3C_r)=\tfrac{1}{44}(36S_{r+2}+28S_{r+1}-12S_r),$$

the last reduction following from (24) and (26). *Hence the coefficient of* x^{r+2} *in the series* (20) *is* $\tfrac{1}{11}(9S_{r+2}+7S_{r+1}-3S_r)2^{r+2}$.

By way of check, we note that $S_1=1$, $S_2=-1$, $S_3=-5$, as shown above. Hence $S_4=-5$, $S_5=1$, by (24). The coefficients of x^3 and x^4 are therefore

$$\tfrac{1}{11}(-45-7-3)2^3=-40,\quad \tfrac{1}{11}(-45-35+3)2^4=-112.$$

Applying Waring's formula (Serret, *Algèbre*, I, page 449) to the cubic $y^3-y^2+y+1=0$, we obtain the explicit result

$$(27)\qquad S_n=\Sigma\frac{(-1)^{\lambda_2+\lambda_3}n(\lambda_1+\lambda_2+\lambda_3)!}{(\lambda_1+1)!(\lambda_2+1)!(\lambda_3+1)!},\qquad \begin{pmatrix}\text{summed for all integers } \gtreqless 0\\ \text{such that } \lambda_1+2\lambda_2+3\lambda_3=n\end{pmatrix}.$$

Second method. Equate expression (21) to $a_0+2a_1x+2^2a_1x^2+\ldots\ldots+2^r a_r x^r+\ldots\ldots$, clear of fractions, and compare coefficients. Then

$$1=a_0,\ 2=-a_0+a_1,\ -3=a_0-a_1+a_2,\ 0=a_0+a_1-a_2+a_3,\ \ldots\ldots,$$

$$0=a_r+a_{r+1}-a_{r+2}+a_{r+3},\quad (r=0,\ 1,\ 2,\ \ldots\ldots).$$

Noting that the determinant of the coefficients of the first n equations is 1, we get

$$(28)\qquad a_n=\left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & & 0 & 1\\
-1 & 1 & 0 & 0 & 0 & 0 & \cdots\cdots & & 0 & 2\\
1 & -1 & 1 & 0 & 0 & 0 & \cdots\cdots & & 0 & -3\\
1 & 1 & -1 & 1 & 0 & 0 & \cdots\cdots & & 0 & 0\\
0 & 1 & 1 & -1 & 1 & 0 & \cdots\cdots & & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots\cdots & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0\\
0 & \cdots\cdots & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0\\
0 & \cdots\cdots & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0\\
0 & \cdots\cdots & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0
\end{array}\right|$$

Expanding according to the elements of the last column, we get

$$(29)\qquad a_n=(-1)^n(D_n-2D_{n-1}-3D_{n-2}),$$

where D_n denotes the minor obtained upon deleting the first row and first column of (28). Adding the first row to the second in D_n, we get

$$D_n = \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \cdots 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 1 & 0 & 0 \cdots \\ 0 & 0 & 1 & -1 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= + \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & -1 & 1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 1 & 0 & 0 \cdots \\ 0 & 1 & 1 & -1 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where the second determinant was reduced by adding its first row to its second. Expanding the third determinant according to its first column,

$$D_n = 2D_{n-3} - D_{n-4}. \tag{30}$$

For $d_r = (-1)^r D_r$, relations (29) and (30) become

$$a_n = d_n + 2d_{n-1} - 3d_{n-2},\ d_n = -2d_{n-3} - d_{n-4}. \tag{31}$$

The second recursion relation is precisely recursion relation (25).

Third method. The existence of the three-term recursion relation, just mentioned, suggested the following method, which aside from its artificiality is perhaps the most satisfactory one of the three. We employ the identity

$$(1-2x+4x^2+8x^3)(1+2x) = 1+16x^3+16x^4.$$

Let ${}_rC_k$ denote the number of combinations of r things k at a time. Then

$$\frac{1}{1+16x^3+16x^4} = \sum_{r=0}^{\infty} (-1)^r \sum_{k=0}^{r} {}_rC_k\, (16x^3)^k (16x^4)^{r-k} = \sum_{r,k} (-1)^r\, {}_rC_k\, 16^r x^{4r-k}.$$

The coefficient of x^n is therefore

$$K_n = \sum_r (-1)^r\, {}_rC_{4r-n} 16^r = \sum_{r=0}^{n'} (-1)^r 16^r\, {}_rC_{n-3r}, \tag{32}$$

where n' is the greatest integer in $n/3$. Now expression (21) equals

$$\frac{(1+4x-12x^2)\,(1+2x)}{(1-2x+4x^2+8x^3)\,(1+2x)} = \frac{1+6x-4x^2-24x^3}{1+16x^3+16x^4}.$$

Hence the coefficients of x^n *in series* (20) *is* $K_n + 6K_{n-1} - 4K_{n-2} - 24K_{n-3}$.